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Inertial migration of a small sphere in linear shear flows

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The motion of a small, rigid sphere in a linear shear flow is considered. Saffman's analysis is extended to other asymptotic cases in which the particle Reynolds number based on its slip velocity is comparable with or larger than the square root of the particle Reynolds number based on the velocity gradient. In all cases, both particle Reynolds numbers are assumed to be small compared to unity. It is shown that, as the Reynolds number based on particle slip velocity becomes larger than the square root of the Reynolds number based on particle shear rate, the magnitude of the inertial migration velocity rapidly decreases to very small values. The latter behaviour suggests that contributions that are higher order in the particle radius may become important in some situations of interest.

1. Introduction

Saffman (1965) considered the lift force on a small sphere in an unbounded, linear shear flow. He assumed that the Reynolds numbers, Re_s and Re_G, were small compared to unity and that $Re_s \ll Re_G^{\frac{1}{2}}$, where $Re_s = v_s d/\nu$ and $Re_G = Gd^2/\nu$; the symbol d denotes the particle diameter, G denotes the velocity gradient, and v_s denotes the magnitude of the particle's slip velocity. Here the term 'slip velocity' refers to the velocity of the sphere centre relative to the undisturbed fluid velocity at that point (resulting, for example, from some non-hydrodynamic external force such as gravity acting on the particle). Harper & Chang (1968) extended Saffman's analysis to arbitrary three-dimensional bodies in a linear shear flow. Drew (1978) performed a similar analysis for a small sphere in a two-dimensional strain flow. The lift force arises because of inertial effects, and the above assumptions imply that the dominant inertial effects arise in the disturbance flow created by the particle at large distances from the particle and that, in this region, the disturbance created by the translation of the sphere can be approximated by a Stokeslet flow.

The main purpose of the present paper is to extend the Saffman analysis to situations in which Re_s is not small compared to $Re_G^{\frac{1}{2}}$. Before providing some motivation for considering this problem, related theoretical work on inertial migration will be reviewed.

Cox & Brenner (1968) considered the lift force on a particle in a wall-bounded flow. They assumed that $Re_l \ll 1$, where Re_l is the Reynolds number based on the distance, l, of the particle's centre from the wall, and that $d \ll l$ so that the particle could be treated as a point force acting on the fluid, to lowest order in d/l. The first of the above two assumptions implies that inertial effects are small compared to viscous effects everywhere in the fluid so that perturbation methods can be used to evaluate the lift force. They derived a general expression for the lift force in terms of a Green's function.

Cox & Hsu (1977) used the theory developed by Cox & Brenner to obtain analytical expressions for the migration velocity of a particle sedimenting parallel to a vertical wall for the cases $Re_G \ll Re_V \ll 1$, where Re_V is a Reynolds number based on a local fluid velocity, $Re_V = Vd/\nu$; $\kappa^2 \ll Re_G/Re_V \ll 1$, where $\kappa = d/l$; and $Re_G/Re_V \gg 1$. In the first case, they ignored fluid shear and considered a sphere sedimenting past a vertical wall in a motionless fluid; they found that the particle is always repelled from the wall in this case. The second case considered by Cox & Hsu is similar to that considered by Saffman in that the shear flow is strong enough that, provided the sphere is far enough from the wall, the particle can be driven toward the wall if it is moving faster than the surrounding fluid. However, the particle Reynolds number, Re_s, is more strongly restricted than in the Saffman analysis because of the requirement that $Re_s \ll Re_l \ll 1$. Cox & Hsu showed that it is possible, in some situations, to achieve an equilibrium in which the shear-induced lift force is equal and opposite to the wall-induced lift force, which is in qualitative agreement with the experimental observations of Segré & Silberberg (1962*a*, *b*) that spheres in a low-Reynolds-number tube flow migrate to a preferred radial position.

Vasseur & Cox (1976) extended the work of Cox & Hsu by considering a sphere moving parallel to the walls of a channel formed by two infinite, parallel walls. They considered the same cases as Cox & Hsu and they obtained the same qualitative types of behaviour.

Vasseur & Cox (1977) were able to remove the restriction $Re_l \ll 1$ for the case of a particle translating through stagnant fluid next to a single planar wall or between two parallel walls. The only restriction on their analysis is that $Re_s \ll 1$. For the case of a particle translating next to a single planar wall, they found that the inertial migration velocity decays as the inverse square of the distance from the wall in the limit $Re_l \gg 1$. On the other hand, in the limit $Re_l \ll 1$, their result reduces to that reported by Cox & Hsu. In all cases, the inertial migration velocity points away from the closest wall. The physical mechanism that causes the migration is that, as the particle translates, it displaces fluid laterally and inertia causes this process to be irreversible at large distances from the particle. If a wall is present at large distances from the particle, it resists the displacement of the fluid and pushes the particle away.

Drew (1988) extended Saffman's analysis by including the effects of a distant wall. Drew assumed that, to zeroth order in inertial effects, the sphere moves parallel to a rigid, flat wall. He further assumed that $d \ll l$ so that the sphere can be treated as a point force acting on the fluid. Finally, he assumed that the sphere was sufficiently far from the wall that inertial effects were of the same order as viscous effects when the distance from the sphere was of order l. Drew found that it is not possible to balance the repulsive lift force due to the wall against the shear-induced lift force for the parameter range in which his treatment is appropriate.

All of the above analyses treat the particle as a point force or, in the case of neutrally buoyant particles, a point force doublet acting on the fluid to leading order. This approach is valid provided that the particle is located at a distance from the closest wall that is large compared to its diameter and that the particle Reynolds numbers are small compared to unity. Relatively little theoretical work has been done on the problem of determining the lift force on a particle that is close to a wall. Leighton & Acrivos (1985) determined the lift on a small sphere that touches a rigid planar surface in the presence of a simple shear flow; they assumed that $Re_G \ll 1$ so that an asymptotic method could be used to derive the lift force. They showed that the lift force points away from the wall and that it varies as the fourth power of the particle radius.

The above analyses are restricted to small particle Reynolds numbers. Auton (1987) has derived an expression for the lift force on a sphere in a weak shear flow of an inviscid fluid. Drew & Lahey (1987) have, independently, suggested expressions for the lift and virtual mass forces on a sphere in an inviscid fluid.

In reviewing the literature, only the most closely related theoretical papers have been discussed. Papers dealing primarily with neutrally buoyant particles, such as the recent work of Schonberg & Hinch (1989), have not been included in the discussion. Leal (1980) has reviewed the literature on inertial migration of particles at low Reynolds numbers up to 1979. There are earlier reviews by Brenner (1966), Goldsmith & Mason (1967), and Cox & Mason (1971).

McLaughlin (1989) reported the results of direct numerical simulations of aerosol motion in a vertical channel flow of turbulent air. The aerosols that deposit develop large streamwise slip velocities in the viscous sublayer adjacent to each channel wall as a result of the large normal component of velocity that the aerosols possess at the edge of the sublayer. The value of Re_G is typically of order 0.04 for the particles that deposit, but Re_s is of order unity. Thus, the formula for the lift force derived by Saffman is not valid for the particles that deposit. One of the goals of the present paper is to provide estimates for the lift force in situations in which Re_s is not small compared to $Re_G^{\frac{1}{2}}$; even though both Reynolds numbers must be assumed to be small compared to unity, it seems plausible that an asymptotic result might give a reasonable estimate for the inertial migration velocity when Re_s is of order unity provided that the asymptotic result is derived for the correct value of $Re_s/Re_G^{\frac{1}{2}}$.

2. Formulation of the problem

It will be assumed that a rigid sphere is located at the origin of a Cartesian coordinate system and that, in the absence of the sphere, the velocity profile is $\boldsymbol{v} = Gx\boldsymbol{e}_3$, where $\boldsymbol{e}_3$ is a unit vector in the z-direction. It is further assumed that the sphere moves along the z-direction at velocity $-v_s\boldsymbol{e}_3$. The objective of the analysis is to derive an expression for the x-component of the forcing acting on the particle. It is convenient to pose the problem in a frame of reference moving with the particle so that the fluid velocity field is time-independent.

Even though Re_G and Re_s are small compared to unity, at sufficiently large distances from the sphere, it is possible to balance inertial effects against viscous effects. Proudman & Pearson (1957) pointed out that this phenomenon is related to the failure of straightforward expansions in the Reynolds number, and they suggested a method of matched asymptotic expansions as a means of incorporating the higher-order effects of inertia in a systematic fashion. Specifically, Proudman & Pearson considered the uniform translation of a rigid sphere through an unbounded, motionless fluid, and they showed how to compute the leading-order Reynolds-number corrections to the Stokes drag coefficient. Briefly, they identified inner and outer regions where the effects of inertia are, respectively, small compared to and comparable in size with viscous effects. If one makes an expansion of the velocity field in Reynolds number in the inner region, it is not possible to satisfy the boundary condition at infinite distance from the sphere. Instead, one must match the large-distance behaviour of the terms of the expansion in the inner region to the small-distance behaviour of the solution of the outer problem. In the outer region, the convective term is well approximated by $v_s\,\partial\boldsymbol{v}/\partial z$, as pointed out by Oseen (1910). To leading order, Proudman & Pearson showed that the sphere can be treated as a point force as far as the flow in the outer region is concerned. The same result will be used in the analysis of the present paper.

Saffman (1965) used a technique similar to that employed by Proudman & Pearson. The primary difference is that, as a result of the assumption $Re_s \ll Re_G^{\frac{1}{2}}$, the convective term in the Navier–Stokes equation is approximated by $Gx\,\partial \boldsymbol{v}/\partial z + Gv_1 \boldsymbol{e}_3$ in the outer region; in this expression, $\boldsymbol{v}$ and v_1 denote the disturbance to the fluid velocity caused by the sphere and the x-component of the disturbance velocity, respectively. Once again, the matching to the inner problem shows that, to leading order, the sphere can be treated as a point force. The primary difference between the present paper and Saffman's work is that Gx will be treated as being of the same order as v_s in the outer region. Thus, in the outer region, the fluid velocity will be approximated by $(Gx+v_s)\,\boldsymbol{e}_3$, and the Navier–Stokes equation will be approximated by

$$(v_s + Gx)\frac{\partial \boldsymbol{v}}{\partial z} + Gv_1 \boldsymbol{e}_3 = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu\nabla^2 \boldsymbol{v} - \frac{\mathbf{F}}{\rho}\delta(\boldsymbol{r}). \tag{2.1}$$

In (2.1), p denotes the pressure in the fluid, $\boldsymbol{r}$ the position vector at a point in the fluid, ρ the fluid density and ν the kinematic viscosity of the fluid. Finally, $\boldsymbol{F}$ denotes the force exerted by the fluid on the particle; to zeroth order in inertial effects, $\boldsymbol{F} = 6\pi\mu a v_s \boldsymbol{e}_3$.

The disturbance flow velocity field is assumed to be incompressible:

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \tag{2.2}$$

and to vanish at large distances from the particle,

$$\boldsymbol{v} = 0, \quad r = \infty. \tag{2.3}$$

Saffman assumed that the Reynolds numbers Re_G and Re_s were small compared to unity and that $Re_s \ll Re_G^{\frac{1}{2}}$. The latter assumption implies that, at distances, r, satisfying $r/d = O(1/(Re_G^{\frac{1}{2}}))$, the convective term is of the same order as the viscous term and that the convective term in (2.1) can be simplified by dropping the term involving v_s. Saffman (1965, including the corrigendum) showed that the lift force on the particle is, for $G > 0$,

$$f_L = 6.46\mu v_s a^2 (G/\nu)^{\frac{1}{2}}. \tag{2.4}$$

For $G < 0$, the sign of the right-hand side of (2.4) must be changed and G is replaced by its magnitude in the argument of the square root. The lift force is related to the migration velocity, v_m, by

$$f_L = 6\pi\mu a v_m. \tag{2.5}$$

Thus, to leading order, the inertial migration velocity predicted by Saffman is, for $G > 0$,

$$v_m = 0.343 a v_s (G/\nu)^{\frac{1}{2}}. \tag{2.6}$$

In the present paper, it will be assumed that the parameter ϵ defined by

$$\epsilon = (G\nu)^{\frac{1}{2}}/v_s, \tag{2.7}$$

or, equivalently, $\epsilon = Re_G^{\frac{1}{2}}/Re_s$, is not necessarily large compared to unity. (However, it will be seen that, if the ratio is too small, higher-order effects that are not considered in the analysis to be presented will become important.) As a consequence, it is not possible to make all of the simplifications that are possible in the limit considered by Saffman. However, it will be shown that the inertial migration velocity can still be expressed in terms of a three-dimensional integral. The matching procedure used by Saffman to determine the inertial migration velocity is valid for the present problem. By eliminating the pressure from (2.1), one obtains an elliptic equation in which the highest derivatives are of biharmonic form, and one can then employ the argument in Saffman (1965, pp. 390–392) to show that the solution of (2.1) contains, to order $1/\nu$ (where ν is treated as a large parameter), a term which appears to the inner expansion as a uniform translation at infinity.

A second point of interest concerns the validity of the linearization of the convective term on which (2.1) is based. One can distinguish three lengthscales in the outer part of the disturbance flow: $L_s = \nu/v_s$; $L_G = (\nu/G)^{\frac{1}{2}}$; and $L_{sG} = v_s/G$. At distances of order L_s (the 'Stokes' lengthscale), the inertial term involving v_s is comparable in size with the viscous term in the Navier–Stokes equation. The inertial terms involving G are comparable in size with the viscous term at distances of order L_G. Finally, the inertial term involving v_s is comparable in size with the inertial terms involving G at distances of order L_{sG}. For $\epsilon \ll 1$, it can be shown that $L_s \ll L_G \ll L_{sG}$, and this implies that, for distances of order L_s, the disturbance flow should be well approximated by the Oseen solution for a sphere translating through a motionless fluid. As a consequence, in this region, the ratio Gx/v is of order ϵ^2/Re_s. Therefore, unless Re_s is small compared to ϵ^2, the neglected convective terms that are quadratic in the disturbance flow will be comparable in size with the gradient terms; since the gradient terms are responsible for the lift, this suggests that the outer expansion may not be valid for computing the lift unless $Re_s \ll \epsilon^2$. Of course, it is not clear *a priori* that the disturbance flow at distances of order L_s will contribute significantly to the lift force, but it will be shown later that this is, in fact, the case.

3. Solution for an unbounded fluid

In this section, the solution of (2.1) for the case of an unbounded fluid will be obtained. For this purpose, it is convenient to introduce the Fourier transforms of the velocity field and the pressure field:

$$\boldsymbol{v} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{u}\, \mathrm{e}^{\mathrm{i}(k_1 x + k_2 y + k_3 z)}\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3, \tag{3.1}$$

and

$$p = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi\, \mathrm{e}^{\mathrm{i}(k_1 x + k_2 y + k_3 z)}\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3. \tag{3.2}$$

By substituting Fourier transforms of the form in (3.1) and (3.2) into (2.1), one can obtain an ordinary differential equation for $\boldsymbol{u}$ which takes the form

$$\mathrm{i}k_3 v_s \boldsymbol{u} = -\mathrm{i}\boldsymbol{k}\Pi/\rho - \nu k^2 \boldsymbol{u} - G u_1 \boldsymbol{e}_3 + G k_3 \frac{\partial \boldsymbol{u}}{\partial k_1} - \frac{F}{8\pi^3 \rho}\boldsymbol{e}_3. \tag{3.3}$$

The pressure satisfies Poisson's equation:

$$\frac{\mathrm{i}k^2 \Pi}{\rho} = -2G k_3 u_1 - \frac{3}{4\pi^2}\nu a v_s k_3. \tag{3.4}$$

The solution of (3.4) is

$$\frac{\Pi}{\rho} = \frac{2\mathrm{i}k_3}{k^2} G u_1 + \frac{3\mathrm{i}k_3}{4\pi^2 k^2}\nu a v_s. \tag{3.5}$$

When (3.5) is used with (3.3), the following equation is obtained for u_1:

$$(\mathrm{i}k_3 v_s + \nu k^2)\, u_1 - \frac{Gk_3}{k^2}\frac{\partial(k^2 u_1)}{\partial k_1} = \frac{(3/4\pi^2)\,\nu a v_s k_1 k_3}{k^2}. \tag{3.6}$$

In order to obtain an algebraic expression for u_1, it is useful to rewrite (3.6) as follows:

$$\frac{\partial\, \mathrm{e}^{\psi} k^2 u_1}{\partial k_1} = -\frac{3}{4\pi^2}\frac{\nu a v_s k_1}{G}\epsilon^{\psi}, \tag{3.7}$$

where

$$\frac{\partial \psi}{\partial k_1} = -\frac{\mathrm{i}v_s}{G} - \frac{\nu k^2}{Gk_3}. \tag{3.8}$$

A suitable solution of (3.8) is

$$\psi = -\frac{\nu k_1^3}{3Gk_3} - \frac{\mathrm{i}v_s k_1}{G} - \frac{\nu(k_2^2+k_3^2)\,k_1}{Gk_3}. \tag{3.9}$$

Provided that $Gk_3 > 0$, the solution of (3.7) is

$$u_1 = -\frac{3}{4\pi^2}\frac{\nu a v_s}{Gk^2}\int_{\infty}^{k_1} \mathrm{e}^{(\psi'-\psi)}\, u\, \mathrm{d}u, \tag{3.10}$$

where ψ' is obtained by substituting u for k_1 in (3.9). The integral in (3.10) can be written in a more convenient form by introducing the dimensionless integration variable, ζ:

$$\zeta = (u-k_1)/k_3. \tag{3.11}$$

The expression for u_1 can now be written in the form

$$u_1 = \frac{3}{4\pi^2}\frac{\nu a v_s k_3}{Gk^2}\int_0^{\infty} \mathrm{e}^{(\psi'-\psi)}\,(\zeta k_3 + k_1)\,\mathrm{d}\zeta, \tag{3.12}$$

where

$$\psi' - \psi = -\frac{\nu}{3G}k_3^2\zeta^3 - \frac{\nu}{G}k_1 k_3 \zeta^2 - \frac{\nu}{G}k^2\zeta - \frac{\mathrm{i}v_s}{G}k_3\zeta. \tag{3.13}$$

The expression for u_1 in (3.12) is valid regardless of the sign of Gk_3. The expression on the right-hand side of (3.13) is identical to the corresponding expression in Saffman's paper, except for the last term.

If the expression for u_1 in (3.12) is substituted into (3.1), the values of the disturbance flow velocity can be calculated at any point in space. As $r = (x^2+y^2+z^2)^{\frac{1}{2}}$ approaches zero, the disturbance flow must approach that of a Stokeslet solution, and, in order to determine the inertial migration velocity, it is necessary to compute the difference between the disturbance flow and a Stokeslet flow and take the limit in which r goes to zero. The Fourier transform of the Stokeslet flow can be obtained from (3.6) by letting k become large:

$$u_1^{\mathrm{s}} = \frac{(3/4\pi^2)\, a v_s k_1 k_3}{k^4}. \tag{3.14}$$

The inertial migration velocity, v_{m}, is given by

$$v_{\mathrm{m}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (u_1 - u_1^{\mathrm{s}})\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3. \tag{3.15}$$

It is possible to perform the integral over the magnitude of $\boldsymbol{k}$ analytically, and, for this purpose, it is useful to perform an integration by parts with respect to ζ:

$$u_1 - u_1^{\mathrm{s}} = \frac{3}{4\pi^2}\frac{\nu a v_{\mathrm{s}}}{G}\int_0^\infty \mathrm{e}^{(\psi'-\psi)}\left[\zeta\left\{\frac{k_3^2}{k^2} - \frac{k_1 k_3(2k_1 k_3 + \zeta k_3^2)}{k^4}\right\} - \frac{(\mathrm{i}v_{\mathrm{s}}/\nu)\, k_1 k_3^2}{k^4}\right]\mathrm{d}\zeta. \quad (3.16)$$

When (3.16) is used with (3.15), and the dimensionless wavevector, $\boldsymbol{q} = (\nu/G)^{\frac{1}{2}}\boldsymbol{k}$ is introduced, it can be shown that

$$v_{\mathrm{m}} = \frac{3}{4\pi^2} a v_{\mathrm{s}}\left(\frac{G}{\nu}\right)^{\frac{1}{2}} I, \quad (3.17)$$

where I is the four-dimensional integral which is defined by

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\left[\zeta\left\{\frac{q_3^2}{q^2} - \frac{q_1 q_3(2q_1 q_3 + \zeta q_3^2)}{q^4}\right\}\cos\left(\frac{q_3\zeta}{\epsilon}\right)\right.$$
$$\left. - \frac{q_1 q_3^2}{\epsilon q^4}\sin\left(\frac{q_3\zeta}{\epsilon}\right)\right]\mathrm{e}^{-(q_3^2\zeta^3/3 + q_1 q_3 \zeta^2 + q^2\zeta)}\,\mathrm{d}\zeta\,\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3. \quad (3.18)$$

The integral over $q = |\boldsymbol{q}|$ can now be performed, and the result is

$$v_{\mathrm{m}} = \frac{3}{2\pi^2} a v_{\mathrm{s}}\left(\frac{G}{\nu}\right)^{\frac{1}{2}} J, \quad (3.19)$$

where

$$J = \int_0^{2\pi}\int_0^1\int_0^\infty\left[\zeta\{s^2 - 2s^2(1-s^2)\cos^2\phi - \zeta s^3(1-s^2)^{\frac{1}{2}}\cos\phi\}\left(\frac{\pi^{\frac{1}{2}}}{4A^3}\right)\left(1 - \frac{B^2}{2A^2}\right)\right.$$
$$\left. - \frac{\pi^{\frac{1}{2}}}{4\epsilon}\frac{B}{A^3} s^2(1-s^2)^{\frac{1}{2}}\cos\phi\right]\mathrm{e}^{-B^2/4A^2}\,\mathrm{d}\zeta\,\mathrm{d}s\,\mathrm{d}\phi. \quad (3.20)$$

In (3.20),

$$A^2 = \tfrac{1}{3}s^2\zeta^3 + s(1-s^2)^{\frac{1}{2}}\cos\phi\,\zeta^2 + \zeta, \quad (3.21)$$

and

$$B = \zeta s/\epsilon. \quad (3.22)$$

In the above equations, $s = \cos\theta$ and ϕ, θ denote the angular coordinates of a spherical coordinate system in Fourier space.

In general, the integrals in (3.20) must be evaluated numerically. First, the asymptotic limits in which $\epsilon \gg 1$ and $\epsilon \ll 1$ will be explored.

The limit $\epsilon \gg 1$ was considered by Saffman. In this case, J reduces to

$$J = \int_0^{2\pi}\int_0^1\int_0^\infty \zeta\left[s^2 - 2s^2(1-s^2)\cos^2\phi - \zeta s^3(1-s^2)^{\frac{1}{2}}\cos\phi\right]\frac{\pi^{\frac{1}{2}}}{4A^3}\,\mathrm{d}\zeta\,\mathrm{d}s\,\mathrm{d}\phi. \quad (3.23)$$

The integrals in (3.23) were evaluated numerically and, to three decimal places, the value of J is 2.255; when this value is substituted in (3.19), one obtains Saffman's (1968) result (see (2.6)). In order to calculate the leading corrections to this value for finite but large values of ϵ, one can expand J in powers of $1/\epsilon$:

$$J = J_0 + J_2/\epsilon^2 + J_4/\epsilon^4 + \dots. \quad (3.24)$$

ϵ	J	ϵ	J
0.025	−0.0000133	0.7	1.2554
0.05	−0.0002845	0.8	1.436
0.1	−0.004658	0.9	1.576
0.15	−0.01458	1.0	1.686
0.2	−0.01247	1.5	1.979
0.25	0.02782	2.0	2.094
0.3	0.1179	5.0	2.227
0.4	0.4076	10.0	2.247
0.5	0.7350	20.0	2.252
0.6	1.0236	∞	2.255

TABLE 1. Values of J for several values of ϵ.

The value of J_0 is 2.255 and J_2 is given by

$$J_2 = \int_0^{2\pi}\int_0^1\int_0^\infty \left[\{s^2-2s^2(1-s^2)\cos^2\phi-\zeta s^3(1-s^2)^{\frac{1}{2}}\cos\phi\}\frac{3\pi^{\frac{1}{2}}}{16A^3}(\zeta^3 s^2) - \tfrac{1}{4}\pi^{\frac{1}{2}}\frac{1}{A^3}(s^2\zeta)(1-s^2)^{\frac{1}{2}}\cos\phi\right]e^{-B^2/4A^2}\,d\zeta\,ds\,d\phi. \quad (3.25)$$

The integrals in (3.25) must be integrated numerically and the result is $J_2 = -0.6463$. Thus, with an error of order $1/\epsilon^4$, J can be approximated by

$$J = 2.255-0.6463/\epsilon^2, \quad (3.26)$$

for $\epsilon \gg 1$.

It can be seen from (3.20) that J and, therefore, v_m/v_s should go to zero as ϵ goes to zero since the exponent becomes large and negative except for values of ζs^2 of order ϵ^2 or ζ of order $1/\epsilon^2$ and $s \gg \epsilon^2$. An important consequence of this fact is that, as ϵ goes to zero, the ratio of the migration velocity to the migration velocity predicted by Saffman's formula approaches zero. The leading behaviour of J in the limit $\epsilon \ll 1$ is

$$J = -32\pi^2\epsilon^5 \ln(1/\epsilon^2). \quad (3.27)$$

The lowest-order corrections to (3.27) are of order ϵ^5. The expression in (3.27) is made up of two identical contributions from the two regions $\zeta s^2 = O(\epsilon^2)$, $s \gg \epsilon^2$ and $\zeta = O(1/\epsilon^2)$, $s \gg \epsilon^2$. In the first region, the leading contribution to J comes from terms of the form $(\zeta s^2)^{\frac{3}{2}}$, where the ratio $\zeta s^2/\epsilon^2$ is treated as order unity. In the second region, the leading contribution to J comes from terms of the form $1/\zeta^{\frac{7}{2}}$, where the product $\zeta\epsilon^2$ is treated as order unity. The coefficients of the above terms were evaluated with the symbolic manipulation program Maxima on Clarkson's School of Engineering Sun computer. If one performs the integrals over ζ and ϕ, one finds terms proportional to $1/s$ that are logarithmically divergent at $s = 0$; this apparent divergence is related to the breakdown of the expansions of the integrand when $s = O(\epsilon^2)$, and, as a consequence, the lower limit of integration must be replaced by a constant of order ϵ^2.

For values of ϵ that are not small compared to unity, it is necessary to evaluate J by numerical integration. The values of J over the range $0.025 \leqslant \epsilon \leqslant 20$ are plotted in figure 1 and displayed in table 1. For comparison, the values of J predicted by the asymptotic formula in (3.26) are also shown in figure 1; for $\epsilon = 1$, the error involved

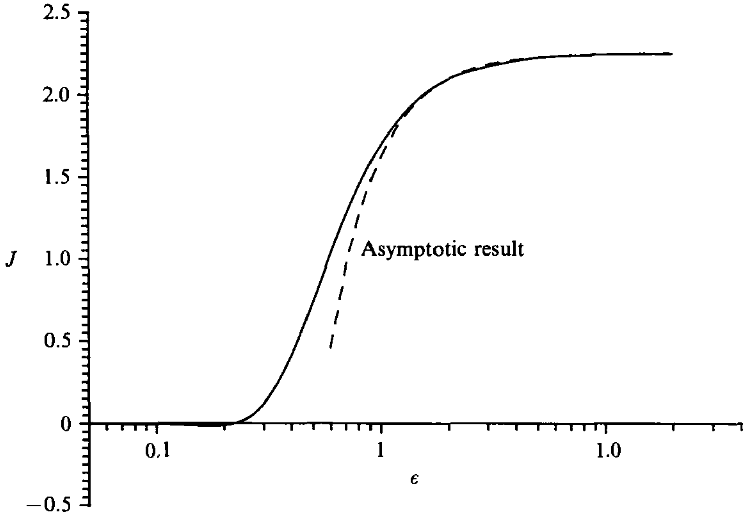

FIGURE 1. The values of J obtained by numerical integration are computed to the values predicted by the asymptotic formula (3.26).

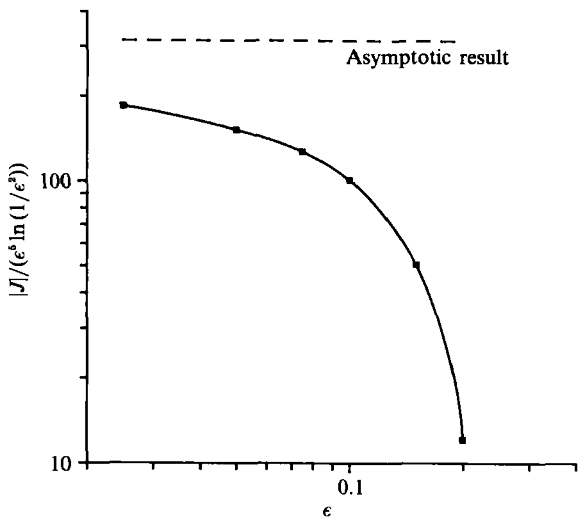

FIGURE 2. The values of $|J|/(\epsilon^5 \ln(1/\epsilon^2))$ obtained by numerical integration are compared to the values predicted by the asymptotic formula (3.27).

in using the asymptotic formula to compute J is 3.4 % and the error involved in using the asymptotic formula to compute the difference between J and Saffman's value for J (2.255) is 7.7 %. The integration was performed in double precision (64 bit) arithmetic on the IBM 3090 computer at the Cornell National Supercomputer Facility. The IMSL routine DQAND was used in the computations; the routine approximates an n-dimensional integral by repeated applications of product Gauss formulae. The routine permits the user to specify the maximum relative and absolute accuracies that are acceptable, and it provides an estimate of the actual error involved in the computation. In order to reduce the amount of computer time, the ζ-integration was broken into several intervals of different sizes. With the exception of the first value, for the values of J reported in table 1, the ζ-intervals were 0 to 10^{-7},

10^{-7} to 10^{-4}, 10^{-4} to 10^{-3}, 10^{-3} to 0.1, 0.1 to 0.5, 0.5 to 1.0, 1.0 to 10, and 10 to 1000; for $\epsilon = 0.025$, the last interval extended to 6000. Since the integrand is weakly singular at $\zeta = 0$, the IMSL routine was not used in the first interval; instead, the following asymptotic formula was used to evaluate the contribution, ΔJ, from the interval 0 to ζ_1 (where ζ_1 was 10^{-7} for the calculations in table 1):

$$\Delta J = \tfrac{1}{5}\pi^{\frac{3}{2}}(\Delta\zeta_1)^{\frac{1}{2}}. \tag{3.28}$$

The effects of the interval sizes and the relative and absolute errors on the computed results were studied, and, with the exception of the value for $\epsilon = 0.025$, the values in table 1 are accurate to within 0.1%; the error for $\epsilon = 0.025$ is 1%. It required approximately two minutes of c.p.u. time to compute each of the values in table 1. As a check on the numerical integration for small values of ϵ, in figure 2 the ratio $|J|/(\epsilon^5 \ln(1/\epsilon^2))$ is compared to the corresponding asymptotic value predicted by (3.27); it can be seen that, although the asymptotic regime has not been reached for $\epsilon = 0.025$, the computed result appears to be approaching the asymptotic limit, and, for $\epsilon = 0.025$, the computed result differs from the asymptotic prediction by a factor smaller than 2.

4. Discussion

The expression for v_m/v_s in (3.19) and the values of J in table 1 are the main results of the present paper. It has been shown that, in general, Saffman's formula overestimates the magnitude of v_m/v_s. For $\epsilon \geqslant 1$, the error involved in using Saffman's formula is at most 25% and the error decreases monotonically as ϵ increases. However, when ϵ is substantially smaller than unity, the overestimate is serious. The sign of the migration velocity predicted by (3.19) is consistent with the Saffman formula for $\epsilon > 0.22$; when both G and v_s are positive, the migration velocity is positive, which implies that a particle would move in such a way that the undisturbed fluid velocity near the particle would increase. However, for $\epsilon < 0.22$, the Saffman formula does not correctly predict even the sign of the migration velocity.

The result for the migration velocity presented in this paper is correct only to lowest order in the particle radius. As pointed out by Saffman (1965), when one carries the inner problem to second order, one obtains a contribution to the lift force that is proportional to the cube of the particle radius and which is independent of the fluid viscosity:

$$f_L^{(2)} = \pi\rho v_s \Omega a^3 - \tfrac{11}{8}\pi\rho v_s G a^3, \tag{4.1}$$

where Ω is the angular velocity of the sphere. When the fluid shear rate, G, vanishes and the particle angular velocity, Ω, is non-zero, the expression for $f_L^{(2)}$ in (4.1) reduces to the lift force derived by Rubinow & Keller (1961) who considered a rotating sphere that translates through a stagnant fluid. The sign of the Rubinow–Keller force is the same as Saffman's first-order lift force. However, if G is not equal to zero, one expects that, for small particle Reynolds numbers, $\Omega = \frac{1}{2}G$ if there is no external torque, and, in this case, $f_L^{(2)}$ has the opposite sign to the first-order lift force. Since the contribution to the migration velocity from the second-order Saffman force is quadratic in the particle diameter, it is formally of higher order than the lowest-order migration velocity predicted by (3.19), which is linear in the particle diameter. However, the present paper shows that the lowest-order migration velocity becomes extremely small when ϵ becomes substantially smaller than unity. Specifically, for $\epsilon \ll 1$, it follows from (3.19), (3.27), and (4.1) that the ratio of the second-order migration velocity to the leading-order contribution to the migration velocity is

$$\frac{v_{\mathrm{m}}^{(2)}}{v_{\mathrm{m}}} = \frac{7}{4608} \frac{Re_{\mathrm{s}}}{\epsilon^4 \ln(1/\epsilon^2)}, \tag{4.2}$$

where it has been assumed that $\Omega = \frac{1}{2}G$. Thus, the second-order term will be larger than the first-order term if $Re_{\mathrm{s}} > 658\epsilon^4 \ln(1/\epsilon^2)$. The condition that the gradient terms in the convective term of (2.1) should be larger than the neglected terms in the Oseen region imposes a weaker (at least in the asymptotic limit as ϵ goes to zero) constraint on the validity of (3.19). It was pointed out in §2 that the ratio Gx/v is of order $\epsilon^2/Re_{\mathrm{s}}$ at distances of order ν/v_{s} and this suggests that terms which are quadratic in the disturbance flow might be comparable with the gradient terms unless $Re_{\mathrm{s}} \ll \epsilon^2$ in the limit $\epsilon \ll 1$. Furthermore, inspection of (3.18) suggests that distances of order ν/v_{s} contribute significantly to the migration velocity for $\epsilon \ll 1$; in this limit, it has been shown in §3 that the dominant contributions to J (and, therefore, I) are from the regions $\zeta s^2 = O(\epsilon^2)$, $s \gg \epsilon^2$ and $\zeta = O(1/\epsilon^2)$, $s \gg \epsilon^2$. Consider the first region; it can be seen from (3.18) that, for $\zeta s^2 = O(\epsilon^2)$, the dominant contribution to I is from values of q of order $1/\epsilon$, and since $q = (\nu/G)^{\frac{1}{2}} k$ and k is the wavenumber, it follows that the contribution from distances of order ν/v_{s} can be significant. As a consequence, it appears that the condition that the gradient terms should be large compared with the terms which are quadratic in the disturbance flow requires that Re_{s} must go to zero more rapidly than ϵ^2 as ϵ goes to zero in order for (3.19) to give a good approximation for the migration velocity. Therefore, at least in the asymptotic limit as ϵ goes to zero, Re_{s} must go to zero faster than $\epsilon^4 \ln(1/\epsilon^2)$ in order for (3.19) to be a good approximation for the migration velocity.

It is difficult to estimate the range of Re_{s} for which (3.19) and the values of J given by table 1 are valid in the interval $0.1 < \epsilon < 1$. Experimental measurements of the lateral migration velocity, v_{m}, over the interval $0.1 < \epsilon < 1$ for a variety of Reynolds numbers in a vertical linear shear flow would be extremely valuable for this purpose. It is possible that, in some problems of practical interest, (4.1) may give a better estimate of observed migration velocities than the lowest-order result.

The problem studied in the present paper involves a moving boundary since the particle is not constrained to move parallel to the undisturbed flow. The results must be understood in the same sense as Stokes flow problems involving moving boundaries; it is assumed that the inertial migration velocity is sufficiently small that the flow is quasi-static.

One situation in which the results of this paper may be of interest is the inertial deposition of aerosol particles from turbulent shear flows onto a flat, rigid surface. For example, consider a vertical flow in a two-dimensional channel formed by two parallel walls. McLaughlin (1989) has presented numerical simulation results for olive oil droplets in a turbulent air flow. At room temperature and pressure, the density ratio for olive oil and air is 763, the kinematic viscosity of air is 0.148 cm^2/s, and the density of air is 1.205 kg/m^3. Consider a flow for which the Reynolds number, based on the channel half-width and the (time-average) centreline velocity, is 2000 (the corresponding pipe flow Reynolds number, based on the diameter and the bulk velocity, is about 6500); if the channel half-width is 1 cm, the friction velocity is 18.8 cm/s, and the 'wall units' (based on the kinematic viscosity and the friction velocity) for length and time are 0.00789 cm and 0.000421 s, respectively; in what follows, a '+' superscript will denote the value of a variable in wall units (e.g. l^+ denotes the distance of the sphere centre from the wall when made dimensionless in

Re_s	v_s (cm/s)	v_s^+	ϵ	First-order Saffman	Second-order Saffman	Present
0.1	8.62	0.459	2.18	−0.0172	0.00173	−0.0161
0.2	17.24	0.918	1.09	−0.0344	0.00173	−0.0268
0.4	34.5	1.84	0.545	−0.0688	0.00173	−0.0266
0.6	51.7	2.75	0.363	−0.103	0.00173	−0.0125
0.8	69.0	3.67	0.272	−0.138	0.00173	−0.00373
1.0	86.2	4.52	0.218	−0.172	0.00173	0.000255

TABLE 2. Values of the inertial migration velocity according to the first- and second-order Saffman formulae and the present paper for olive oil aerosol droplets in a vertical channel flow of turbulent air.

terms of the friction velocity and the kinematic viscosity). For a particle having a relaxation time equal to 2 wall units, the radius is 8.60 μm or 0.109 wall units. For droplets of this size, the gravitational settling velocity is extremely small and, for that reason, it is unimportant whether the air flow is upward or downward. In table 2, the inertial migration velocities predicted by the first-order Saffman formula (2.4), the second-order Saffman formula (4.1), and (3.19) are given for olive oil droplets with particle Reynolds numbers, based on v_s and the particle diameter, between 0.1 and 1.0; the values of the migration velocity are given in wall units (i.e. one must multiply them by the friction velocity in order to obtain the value in SI units). The sign convention is that positive migration velocities point toward the centre of the channel. In the simulations, the slip velocity, v_s, is caused by the turbulent fluctuations in the core of the channel and the inertia of the droplets; the droplets are thrown into the viscous sublayer by the turbulent eddies and, as a result of the large normal gradient of the streamwise component of the undisturbed fluid velocity and the droplets' inertia, they acquire a large streamwise slip velocity. In computing the migration velocity due to the second-order Saffman force, it was assumed that the angular velocity of the sphere was $\frac{1}{2}G$. It can be seen that, for $Re_s = 0.4$, the inertial migration velocity predicted by (3.19) is only 39% of the value predicted by the first-order Saffman formula, and, as Re_s increases beyond this value, the migration velocity decreases to very small values, eventually becoming smaller than the value predicted by the second-order Saffman formula. The values of the migration velocity in table 2 must be interpreted with caution as Re_s approaches unity since none of the three formulae can claim validity in this limit. It can be seen from table 1 that there is considerable uncertainty about the magnitude and even the sign of the migration velocity for Re_s of order unity and $\epsilon < 0.5$.

In the above discussion, wall effects have been ignored. One can use the Vasseur–Cox (1977) theory to obtain an estimate of the inertial migration velocity when the fluid can be treated as stagnant to a first approximation (i.e. $Re_s \gg Re_G^{\frac{1}{2}}$ and the particle is close enough to the wall that wall effects are significant). In table 3, the inertial migration velocity is given for olive oil droplets with the same range of particle Reynolds numbers considered in table 2. The distance from the wall, l, is chosen so that $Re_l = 1$. Cherukat & McLaughlin (1990) have shown that the Vasseur–Cox result is a good estimate for v_m even when Re_s is of order unity, and even when the sphere's centre is only a few diameters from the wall; in this case, v_m^+ can be of order 0.1. The results of McLaughlin (1989) indicate that inertial migration velocities of this size could have a significant effect on the deposition of aerosols. It should also be noted that the migration velocities predicted by the Vasseur–Cox

Re_s	v_s (cm/s)	v_s^+	l (cm)	l^+	$l^+/2a^+$	v_m^+
0.1	8.62	0.459	0.0172	2.18	10.0	0.00179
0.2	17.24	0.918	0.00860	1.09	5.0	0.00715
0.4	34.5	1.84	0.00430	0.545	2.5	0.0286
0.6	51.7	2.75	0.00287	0.363	1.67	0.0644
0.8	69.0	3.67	0.00215	0.272	1.25	0.115
1.0	86.2	4.59	0.00172	0.218	1.00	0.179

TABLE 3. Values of the wall-induced inertial migration velocity in wall units predicted by the Vasseur–Cox theory for olive oil aerosol droplets in a vertical channel flow of turbulent air.

formula are positive (i.e. away from the closest wall) while the Saffman formula predicts negative values for the migration velocity. For the smallest values of ϵ, the first-order Saffman formula predicts that the shear-induced lift is comparable in size with and opposite in direction to the wall-induced lift predicted by the Vasseur–Cox formula, while (3.19) predicts that shear-induced lift is negligible in comparison with the wall-induced lift. Cherukat (1990) has conducted measurements of lateral migration velocities for plastic balls sedimenting in a vertical laminar channel flow for $0.055 < \epsilon < 0.374$ and $0.31 < Re_s < 2.38$. The balls were released at roughly 3.5 ball diameters from the closest channel wall; at this distance, the first-order Saffman formula predicts migration velocities that are roughly equal and opposite to the migration velocity predicted by the Vasseur–Cox formula, while (3.19) predicts that the shear-induced lift should be negligible compared with the wall-induced lift. Cherukat found that the migration velocity was always toward the centre of the channel and that it was in reasonable agreement with the Vasseur–Cox formula.

5. Conclusion

The result for the inertial migration velocity in (3.19) is valid for arbitrary values of the parameter ϵ defined by (2.7) provided that Re_s is sufficiently small compared to unity; in the asymptotic limit as ϵ goes to zero, this restriction is that $Re_s \ll 658\epsilon^4 \ln(1/\epsilon^2)$. Unfortunately, there are no experimental measurements that can be used to determine the range of Re_s for which the asymptotic formula in (3.19) is valid over the range $0.1 < \epsilon < 1$; this interval is particularly interesting because of the rapid variations of the ratio v_m/v_s and the fact that the ratio changes sign within the interval.

For $\epsilon \geqslant 1$, the asymptotic approximation in (3.26) gives values for the integral J that appears in (3.19) which are accurate to within 3.4%. For $\epsilon \ll 1$, J can be approximated by the asymptotic expression in (3.27). However, as noted above, the range of Re_s for which (3.19) is valid is more limited for small values of ϵ than for large values.

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